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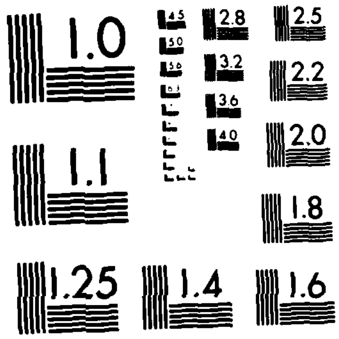

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The Symmetry Group and Exponents of

Operator Stable Probability Measures

by

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THE SYMMETRY GROUP AND EXPONENTS OF OPERATOR STABLE PROBABILITY MEASURES

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Abstract

There exist exponents of an operator stable measure which commute with every operator in the measure's symmetry group. These exponents together with a new norm lead to some simplifications in the representation of the Lévy measure.

Keywords: Operator-stable laws, multivariate symmetric stable distributions, multivariate stable laws.

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0. Introduction

An operator-stable (OS) probability measure μ on a normed finite-dimensional real vector space V is the limit distribution of operator normed and centered sums of a sequence of i.i.d. random vectors in V. The classical stable laws on $\mathbb{R}^1$ are a special case. If μ is full and operator stable, then μ is infinitely divisible so if $\hat{\mu}$ is the ch.f. of μ, then for $t>0$, $\hat{\mu}^t$ is the ch.f. of an infinitely divisible measure μ^t. The role of the index in the one-dimensional case is played by an invertible linear operator B on V called the exponent of μ. If we define $t^B = \exp\{(\mathrm{Ln}\, t)B\} = \sum_{j=0}^{\infty} \frac{(\mathrm{Ln}\, t)^j}{j!} B^j$, then B is an exponent for μ if

$$\mu^t = t^B \mu * \delta(b(t)) \ , \ t>0, \tag{1}$$

where $\delta(b(t))$ is the unit mass at $b(t) \in V$ and $t^B\mu = \mu t^{-B}$. In [7] it was proved that full OS distributions always have at least one exponent.

An exponent of a full OS law μ determines much of its structure. (See [2] and [7] for the results which are now described.) In general μ has both a Gaussian component μ_g and a Poisson component μ_p. These components are concentrated on independent subspaces determined by the exponent B. To be precise let f(x) denote the minimal polynomial of B. Then $f(x) = g(x)h(x)$ where the roots of g have real parts equal to ½ while those of h have real parts greater than ½. The Gaussian component μ_g is concentrated on $V_g = \mathrm{kernel}(g(B))$ while μ_p is concentrated on $V_p = \mathrm{kernel}(h(B))$. Furthermore $V = V_g \oplus V_p$, μ_g and μ_p are full and OS on V_g and V_p respectively. The exponents of μ_g and μ_p are the restrictions of B to V_g and V_p respectively. Now let M denote the Lévy measure of μ. The exponent determines a major part of the structure of M. From (1) upon noting that $t \cdot M$ is the Lévy measure of μ^t and that $t^B M = Mt^{-B}$ is the Lévy measure of $t^B\mu$, one sees that $t \cdot M = t^B M$. This fact can be used to show that if A is a Borel subset of V_p, then

$$M(A) = \int_L M_x(A) K(dx) \tag{2}$$

where K is a finite measure on a Borel subset L of the unit sphere U in V_ρ and M_x is concentrated on the single orbit $\{t^B x: t > 0\}$ determined by x. The Lévy measure M_x also satisfies the condition that $t \cdot M_x = t^B M_x$ and as a result,

$$M_x\{t^B x: t > s\} = 1/s, \ s > 0,$$

(i.e. $M_x(A) = \int_0^\infty I_A(t^B x) t^{-2} dt$). From (2) it follows that the support of M is the union of orbits of t^B. Each orbit begins at the origin and extends to infinity (i.e. $\lim_{t\to 0} t^B x = 0$ and $\lim_{t\to\infty} \|t^B x\| = \infty$). The shape of these orbits is determined by the exponent B. In particular cases orbits can be straight lines ($B = \lambda I$), half of a parabola ($B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $V = \mathbb{R}^2$), or spirals (e.g. $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = I + Q$ where $Q + Q^* = 0$ so t^Q is a rotation). The expression for M_x above shows that the tail behavior of M along orbits is determined by B. The measure K assigns weights to the orbits and determines which orbits are included in the support of M. Together B and K determine M. But, in general B and K are not unique. Is there a reasonable way to choose a particular exponent and measure K? The set of exponents depends on the amount of symmetry possessed by μ. Call a linear operator A on V a symmetry of μ if for some $a \in V$, $\mu = A\mu * \delta(a)$. It is natural to expect that a symmetry of μ should take orbits into orbits while leaving K invariant. (See Theorem 7 below.) In particular, if $BA = AB$, then $At^B x = t^B Ax$ (since t^B is a power series in B) so orbits are taken by A into orbits. Furthermore the requirement that B commutes with every symmetry tends to pick out exponents with nice properties whenever possible. (See Theorems 4 and 5.)

Example. Suppose that μ is the standard Gaussian measure on $\mathbb{R}^d$. If X and Y are i.i.d. μ, the measure corresponding to $X + Y$ is $\mu * \mu = 2^{\frac{1}{2}}\mu$. One suspects (and easily verifies) that $\frac{1}{2}I$ is an exponent for μ. Suppose that S is a skew operator, that is, that $S + S^* = 0$. For each $t > 0$, t^S is orthogonal and so $t^S\mu = \mu$, i.e. t^S is a symmetry of μ. It follows that $\frac{1}{2}I + S$ is also an exponent for μ, for any skew operator S (see Theorem 1 below). Thus operator stable measures may have many

exponents; the number of exponents depends on the size of the collection of symmetries of μ. Does an operator stable measure have a "simplest" exponent?

A lemma of Schur's ([6], p. 173) suggests a possible answer. This lemma says: "Let F be a family of linear operators on a Hilbert Space H and suppose that the only closed subspaces which are invariant under every operator in F are {0} and H. If A is a self-adjoint linear operator on H that commutes with every operator in F, then A=cI for some scalar c." (As usual, I denotes the identity operator.) Schur's Lemma suggests that the "simplest" exponent would be one which commutes with a large collection of operators. In this example, $\frac{1}{2}I$ is the only exponenet of μ which commutes with every symmetry of μ. We will show below that there is always an exponent of μ which commutes with all the symmetries of μ. (Theorem 2)

Our results on commuting exponents are applied to simplify the representation of the Lévy measure of an OS law in section 3. There we define a new norm. The unit sphere relative to this norm plays the role of L above. The corresponding mixing measure K does not depend on the choice of an exponent (Theorem 6). This representation provides a simple relationship between the symmetries of μ and those of K. These results complement those of Kucharczak [5], Jurek [3], and Hudson-Mason [2].

1. Preliminaries

Let μ be a full OS probability measure on a finite dimensional real vector space V. GL(V) denotes the set of all invertible operators on V. For $A \in GL(V)$, we define $A\mu = \mu \circ A^{-1}$. Two groups of interest in connection with μ are the symmetry group

$$S(\mu) = \{A \in GL(V): A\mu * \delta(a) = \mu \text{ for some } a \in V\}$$

and

$$G = \{A \in GL(V): \text{for some } t > 0, \text{ for some } a \in V, \ \mu^t = A\mu * \delta(a)\}.$$

It is known that $S(\mu)$ is a compact, normal subgroup of G. For any closed group H, TH will denote the tangent space of H. Thus $A \in TH$ if and only if $A = \lim_{n\to\infty} (H_n - I)/d_n$ where $\{H_n\} \subset H$ and $\{d_n\}$ is a real null sequence. We recall that the exponential maps TH onto the connected component of I in H. CH will denote the center of H, that is, those elements of H which commute with every element of H. Recall that CH is a subgroup of H.

The collection of exponents of μ, denoted $E(\mu)$, is the set of all operators for which (1) holds. The following result gives a basic fact about exponents.

Theorem 1. Let $B \in E(\mu)$. Then

(i) Every eigenvalue of B has real part $\geq \frac{1}{2}$,

(ii) $E(\mu) = B + TS(\mu)$.

For a proof of this result see [1] and [7].

2. Commuting Exponents

In this section we investigate the existence of an exponent which commutes with every operator in $S(\mu)$. Such exponents will be called commuting and the collection of commuting exponents will be denoted by $E_c(\mu)$.

Proposition 1. Let $A \in S(\mu)$ and $B \in E(\mu)$. Then $ABA^{-1} \in E(\mu)$. Moreover, if $S(\mu)$ is discrete, the unique exponent B is commuting.

Proof. We have $A\mu = \mu * \delta(a)$ and

$$(A\mu)^t = A\mu^t = A(t^B \mu * \delta(b(t))) = At^B \mu * \delta(Ab(t)) = t^{ABA^{-1}} (A\mu) * \delta(Ab(t)).$$

Hence

$$\mu^t = t^{ABA^{-1}} \mu * \delta(Ab(t) - ta + t^{ABA^{-1}} a)$$

and $ABA^{-1} \in E(\mu)$. Now if $S(\mu)$ is discrete, $TS(\mu) = 0$ and B is the unique exponent. Thus $ABA^{-1} = B$ and B is commuting. (QED)

The following example shows that not all exponents are commuting.

Example. Let μ be the symmetric Cauchy distribution on R^2. Then $I \in E(\mu)$ and $S(\mu)$ is the full orthogonal group. Hence $TS(\mu)$ consists of the skew symmetric operators. By Theorem 1, $E(\mu) = I + TS(\mu)$ so $B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ is an exponent. Also $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in S(\mu)$. A direct computation shows that $AB \neq BA$. Furthermore, A does not map orbits into other orbits.

The main result of this section is that commuting exponents always exist.

Theorem 2. $E_c(\mu)$ is non-empty.

Proof. Let H be a Haar probability measure on the compact group $S(\mu)$, and let $B \in E(\mu)$. Define

$$M = \int_{S(\mu)} sBs^{-1}dH(s).$$

Since $E(\mu)$ is closed and convex by Theorem 1 and closed under conjugation by elements of $S(\mu)$ by Proposition 1, $M \in E(\mu)$. If $A \in S(\mu)$, then by the invariance property of Haar measure

$$AMA^{-1} = \int_{S(\mu)} AsBs^{-1}A^{-1}dH(s) = \int_{S(\mu)} (As)B(As)^{-1}dH(s) = \int_{S(\mu)} sBs^{-1}dH(s) = M.$$

Thus $M \in E_c(\mu)$. (QED)

The collection of all commuting exponents is characterized in our next result.

Theorem 3. Suppose $B \in E_c(\mu)$. Then $E_c(\mu) = B + TCS(\mu)$.

Proof. Assume $B \in E_c(\mu)$. Using the relation between groups and their tangent spaces one readily verifies the equivalence of the following statements.

(i) $\tilde{B} \in E_c(\mu)$,

(ii) $\tilde{B} - B \in TS(\mu)$ and $\tilde{B} - B$ commutes with every element of $S(\mu)$.

(iii) For all t, $\exp\{t(\tilde{B} - B)\} \in CS(\mu)$, and

(iv) $\tilde{B} - B \in TCS(\mu)$. (QED)

Corollary. $E_c(\mu) = E(\mu)$ if and only if $TS(\mu) = TCS(\mu)$.

We now examine the extent to which the structure of a commuting exponent is determined by the "size" of $S(\mu)$.

Theorem 4. Let $B \in E_c(\mu)$. If the only proper subspace of V invariant under $S(\mu)$ is 0, then $B = \lambda I + WQW^{-1}$, where W is positive definite and Q is skew symmetric. Furthermore either $Q = 0$ or $Q^2 = WQ^2W^{-1} = -\beta^2 I$ for some $\beta > 0$.

Proof. Since $S(\mu)$ is compact, there is a positive definite operator W and a closed subgroup G of the orthogonal group such that

$$S(\mu) = WGW^{-1}.$$

It follows that $S(W^{-1}\mu) = G$. Since $B \in E_c(\mu)$, $B_0 \equiv W^{-1}BW \in E_c(W^{-1}\mu)$. Write $B_0 = B_1 + B_2$ where $B_1 = \frac{1}{2}(B_0 + B_0^*)$ is self-adjoint and $B_2 = \frac{1}{2}(B_0 - B_0^*)$ is skew-symmetric. Since $B_0 \in E_c(W^{-1}\mu)$, $AB_0 = B_0A$ for $A \in G$. Take adjoints to see that $B_0^*A^* = A^*B_0^*$ for $A \in G$. But every operator in G is orthogonal so $G = \{A^* : A \in G\}$. Thus

$$AB_0^* = B_0^*A, \quad A \in G.$$

It follows that every operator in G commutes with B_1 which is self-adjoint. Now by hypothesis the only proper subspace of V invariant under $S(\mu)$ and hence under G is 0. By Schur's Lemma, $B_1 = \lambda I$ for some real number λ. Now consider B_2. Since B_2 is skew-symmetric, it is normal and thus its minimal polynomial is the product $p_1(x), \ldots, p_k(x)$ of distinct irreducible polynomials. If $k > 1$, then $\ker p_1(B_2)$ is a proper subspace of V which is invariant under G contrary to our hypothesis. Thus $k = 1$ and the minimal polynomial of B_2 is either x or $x^2 + \beta^2$ for some $\beta > 0$. (A skew-symmetric operator has purely imaginary eigenvalues). If it is x, then $B_2 = 0$; otherwise, $B_2^2 = -\beta^2 I$. From $B_0 = B_1 + B_2 = \lambda I + B_2$, we obtain upon setting $Q = B_2$,

$$W^{-1}BW = \lambda I + Q$$

or $B = \lambda I + WQW^{-1}$. Finally $B \in E(\mu)$ so the real part of every eigenvalue of B is not less than $\frac{1}{2}$, i.e. $\lambda \geq \frac{1}{2}$. (QED)

Corollary. If in addition to the hypothesis of the theorem, either B is diagonalizable or dim V is odd, then $B = \lambda I$.

Proof. First suppose B is diagonalizable. Let v be an arbitrary eigenvector of B so $Bv = \lambda_0 v$ for some real λ_0. By Theorem 4, $B = \lambda I + WQW^{-1}$ so v is an eigenvector of WQW^{-1}. In particular $WQW^{-1}v = (\lambda_0 - \lambda)v$. Hence $(WQW^{-1})^2 v = (\lambda_0 - \lambda)^2 v$. But $WQW^{-1} = 0$ or $(WQW^{-1})^2 = -\beta^2 I$. In either case it follows that $\lambda_0 = \lambda$. Since B is diagonalizable, $B = \lambda I$. Now suppose $\dim V$ is odd. Since Q is skew symmetric,

$$\det Q = \det Q^* = \det(-Q) = -\det Q,$$

so Q is singular. Hence $Q^2 \neq -\beta^2 I$ and therefore $Q = 0$. (QED)

A slight refinement of the preceding theorem is given in

Theorem 5. Suppose $B \in E_c(\mu)$ has p real eigenvalues $\lambda_1, \ldots, \lambda_p$ with corresponding eigenvectors $v_1, \ldots, v_p$. If $\{Av_i : A \in S(\mu)\ 1 \le i \le p\}$ spans V, then B is diagonalizable with spectrum $\{\lambda_1, \ldots, \lambda_p\}$. Thus if $\lambda_1 = \ldots = \lambda_p = \lambda$, $B = \lambda I$.

Proof. For $A \in S(\mu)$, $BAv_i = ABv_i = \lambda_i Av_i$, so Av_i is an eigenvector of B with eigenvalue λ_i. Hence there is a basis of V consisting of eigenvectors of B and so B is diagonalizable. (QED)

Corollary. In R^2 if $B \in E_c(\mu)$ and if there is a reflection $A \in S(\mu)$, then B is self-adjoint.

Proof. Select orthonormal vectors v_1 and v_2 so that $Av_1 = v_1$ and $Av_2 = -v_2$. Then $ABv_1 = Bv_1$ and $ABv_2 = -Bv_2$, so $Bv_1 = \lambda_1 v_1$ and $Bv_2 = \lambda_2 v_2$ where λ_1 and λ_2 are real. (QED)

3. The Lévy measure

In this section we discuss the relationship between commuting exponents and the representation of the Lévy measure of μ. Since μ is infinitely divisible, one can write the characteristic function of μ in the canonical form

$$\hat{\mu}(y) = \exp\{i\langle y,a\rangle - \tfrac{1}{2}\langle y,\Sigma y\rangle + \int \psi(x,y)M(dx)\}$$

where $a \in V$, Σ is a non-negative definite self-adjoint operator, M is a

σ-finite measure satisfying

$$\int_V ||x||^2 \wedge 1 M(dx) < \infty,$$

and

$$\psi(x,y) = \exp\{i<x,y>\} - 1 - \frac{i<x,y>}{1+<x,x>}\}.$$

For OS measures it has been shown that one can further decompose the Lévy measure M as follows. For an exponent B of μ set $L_B = \{||x||=1 \text{ and } ||t^B x||>1 \text{ for all } t>1\}$ and define the mixing measure K_B on the Borel subsets A of L_B by

$$K_B(A) = M(\{t^B x: x\in A,\ t \geq 1\}).$$

Thus K_B assigns mass to the particular orbits $\{t^B x: t>0\}$. Note that both L_B and K_B depend on the choice of exponent B. In terms of K_B the Lévy measur M is given by

$$M(S) = \int_{L_B} \int_0^\infty I_S(t^B x) t^{-2} dt dK_B(x) \tag{3}$$

(See [2] and [3].) It was necessary to introduce the subset L_B of U since for some exponents, orbits may intersect the unit sphere more than once.

We now introduce a new norm $|||\cdot|||$ which depends on the particular OS law but not on the choice of exponent. The unit sphere $U'=\{v: |||v|||=1\}$ induced by this norm will intersect each orbit once and so may play the role of L_B. As above we define a mixing measure K on the Borel subsets A of U' by $K(A)=M\{t^B x: x\in A,\ t\geq 1\}$. This measure K also does not depend on the choice of exponent and the representation (3) of the Lévy measure M in terms of K is still valid. The new norm leads to a system of "polar" coordinates with nice properties. (cf. Jurek [4]).

For $x\in V$, and $B\in E(\mu)$ define $|||x||| = \int_0^1 \int_{S(\mu)} ||gt^B x|| H(dg) t^{-1} dt$ where H again denotes Haar measure on $S(\mu)$ and $||\cdot||$ is the original norm on V.

Proposition 3. If μ is full and OS on V, then

(i) $|||\cdot|||$ does not depend on the choice of $B\in E(\mu)$,

(ii) $|||\cdot|||$ is a norm on V,

(iii) for $A\in S(\mu)$, $|||Ax||| = |||x|||$,

(iv) $t \to |||t^B x|||$ is strictly increasing on $(0,\infty)$ for each $x\neq 0$, and

(*v*) the map $\Phi_B: U' \times (0,\infty) \to V \setminus \{0\}$ defined by $\Phi_B(x,t) = t^B x$ is a homeomorphism when $U' \times (0,\infty)$ has the product topology.

Proof. (*i*). Let $B \in E(\mu)$ and let $B_0 \in E_c(\mu)$. By Theorem 1, $B - B_0 \in TS(\mu)$ so for all $t > 0$, $B_0 t^{B-B_0} = t^{B-B_0} B_0$. Differentiate to see that B_0 commutes with $B - B_0$ and consequently that $t^B = t^{B-B_0} t^{B_0}$. For $x \in V$, use the invariance property of Haar measure to obtain the equalities

$$
\begin{aligned}
|||x|||_B &= \int_0^1 \int_{S(\mu)} \|gt^B x\| \, t^{-1} H(dg)\,dt \\
&= \int_0^1 \int_{S(\mu)} \|gt^{B-B_0} t^{B_0} x\| \, t^{-1} H(dg)\,dt = |||x|||_{B_0}.
\end{aligned}
$$

This proves (*i*) and allows us to omit the subscript B.

(*ii*) This is obvious.

(*iii*) Let $A \in S(\mu)$. By (*i*)

we may assume that $B \in E_c(\mu)$. Then

$$
\begin{aligned}
|||Ax||| &= \int_0^1 \int_{S(\mu)} \|gt^B Ax\| \, t^{-1} H(dg)\,dt \\
&= \int_0^1 \int_{S(\mu)} \|gAt^B x\| \, t^{-1} H(dg)\,dt = |||x|||.
\end{aligned}
$$

(*iv*) Suppose that $0 < r < s$. Then

$$
\begin{aligned}
|||r^B x||| &= \int_0^1 \int_{S(\mu)} \|g(tr)^B x\| \, t^{-1} H(dg)\,dt \\
&= \int_0^r \int_{S(\mu)} \|gu^B x\| \, u^{-1} H(dg)\,du \\
&< \int_0^s \int_{S(\mu)} \|gu^B x\| \, u^{-1} H(dg)\,du = |||s^B x|||.
\end{aligned}
$$

(*v*) It follows from (*iv*) that Φ_B is one-to-one. Since every point in $V \setminus \{0\}$ lies on some orbit, Φ_B is "onto". The continuity of Φ_B is well-known and easily checked. To show Φ_B^{-1} is continuous write $\Phi_B^{-1}(x) = (\ell(x), \zeta(x))$ so that $\ell(x) \in U'$, $\zeta(x) > 0$ and $x = \zeta(x)^B \ell(x)$. Suppose that $x_n \to x \neq 0$. Assume some subsequence

$\zeta(x_{n'})$ tends to infinity. Then since the eigenvalues of B have positive real parts, $\|x_{n'}\| = \|\zeta(x_{n'})^B \ell(x_{n'})\| \to \infty$ contrary to the convergence of x_n. It follows that $(\ell(x_n), \zeta(x_n))$ is a bounded sequence in $U' \times (0,\infty)$. Let $(\ell(x_{n'}), \zeta(x_{n'}))$ be any convergent subsequence and let $(\ell_0, \zeta_0) = \lim(\ell(x_{n'}), \zeta(x_{n'}))$. Then

$$x = \lim x_{n'} = \lim \zeta(x_{n'})^B \ell(x_{n'}) = \zeta_0^B \ell_0.$$

Since Φ_B is one-to-one, $\zeta(x) = \zeta_0$, and $\ell(x) = \ell_0$. Thus every convergent subsequence of $(\ell(x_n), \zeta(x_n))$ has the same limit, namely $(\ell(x), \zeta(x))$. This proves that Φ_B^{-1} is continuous. (QED)

The proof that Φ_B^{-1} is continuous was given above for the sake of completeness, c.f. [4].

Part (ii) of Proposition 3 implies that each orbit intersects U' exactly once. The fact that U' is closed and that Φ_B is a homeomorphism is useful in proving weak convergence results.

Theorem 6. Let μ be full OS with Lévy measure M and let $B \in E(\mu)$. *Let F and E be any Borel subsets of* $V \setminus \{0\}$ and U' respectively. Then

$$M(F) = \int_{U'} \int_0^\infty I_F(s^B x) s^{-2} ds K(dx) \tag{4}$$

where K is a finite Borel measure on U' and

$$K(E) = M\{t^B x : x \in E, t \geq 1\} \tag{5}$$

The measure K does not depend on the choice of $B \in E(\mu)$.

Proof. The proof of (4) and (5) is similar to that of (3) in [2] or [3] and is therefore omitted.

The proof that K does not depend on the choice of exponent will involve an easy lemma.

Lemma 3.1 Let $g \in S(\mu)$ and $B \in E(\mu)$. If $gB = Bg$, then $gK_B = K_B$.

Proof. Let D be any Borel subset of U'. Then

$$\begin{aligned} gK_B(D) &= K_B(g^{-1}(D)) \\ &= M\{t^B x: x \in g^{-1}(D),\ t \geq 1\} \\ &= M\{t^B g^{-1} x: x \in D,\ t \geq 1\} \\ &= M(g^{-1}\{t^B x: x \in D,\ t \geq 1\}) \\ &= (gM)(\{t^B x: x \in D,\ t \geq 1\}). \end{aligned}$$

But $g \in S(\mu)$ and hence $gM = M$. Thus

$$gK_B(D) = M\{t^B x: x \in D,\ t \geq 1\} = K_B(D). \qquad \text{(QED)}$$

Now let A be any Borel subset of $V\backslash\{0\}$. Then if $B \in E(\mu)$

$$\begin{aligned} M(A) &= \int_0^\infty \int_{U'} I_A(t^B x) t^{-2}\, K_B\,(dx)\, dt \\ &= \int_0^\infty K_B\,((t^{-B}A) \cap U')\, t^{-2} dt. \end{aligned}$$

Let $B_0 \in E_c(\mu)$. It suffices to prove that $K_B = K_{B_0}$. So let D be any Borel subset of U', and put $A = \{s^B x: x \in D,\ s \geq 1\}$. Then

$$(t^{-B}A) \cap U' = \begin{cases} \phi \text{ if } t < 1 \\ D \text{ if } t \geq 1. \end{cases}$$

Hence

$$K_B(D) = M(A) = \int_0^\infty K_{B_0}((t^{-B_0}A) \cap U')t^{-2}dt.$$

But $B_0 \in E_c(\mu)$ so B_0 commutes with $B-B_0$. Furthermore $t^{B-B_0} \in S(\mu)$ and $t^{B-B_0}U' = U'$. It follows from Lemma 3.1 that

$$K_{B_0}((t^{-B}A) \cap U') = K_{B_0}((t^{B-B_0}(t^{-B}A)) \cap U')$$

$$= K_{B_0}((t^{-B_0}A) \cap U').$$

Therefore,

$$K_B(D) = \int_0^\infty K_{B_0}((t^{-B}A) \cap U')t^{-2}dt.$$

$$= \int_1^\infty K_{B_0}(D)t^{-2}dt = K_{B_0}(D). \qquad \text{(QED)}$$

Remark. There is a converse to Theorem 6. If B is an OS exponent, and if K is a finite Borel measure on $U' \cap V_p$, then the measure M defined by

$$M(F)= \int_{U'} \int_0^{\infty} I_F(s^B x)s^{-2} ds K(dx)$$

is the Levy measure of some OS law with exponent B. Again see [2] or [3]. In [7] Sharpe characterized the set of OS exponents, i.e. those operators which are the exponent of some OS law.

We now consider the relationship between S(K), the symmetry group of the measure K in Theorem 6, and $S(\mu)$.

Theorem 7. Let μ be a full OS measure on V. Then $S(\mu) \subset S(K)$.

Proof. Let $A \in S(\mu)$. Since by Proposition 3, $|||Ax||| = |||x|||$, $AU'=U'$. Since K does not depend on the choice of an exponent, we may assume $B \in E_c(\mu)$. Then $S(\mu) \subset S(K)$ follows from Lemma 3.3. (QED)

The following example shows that even if an OS measure μ has no Gaussian component, if the original norm on V is used and if M is defined as in (3), then S(K) may be much larger than $S(\mu)$ even though K is full. (To see that in this example μ has no Gaussian component, note that no eigenvalue of B has real part equal to ½.)

Example. Take $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then L_B is the unit circle in R^2. Let K be the Lebesgue measure on the circle. Then K is full and S(K) is the orthogonal group. Define M (and hence μ) in terms of K and B using equation (3). Then μ is a full OS measure with $B \in E(\mu)$ (see [2]). We now find $S(\mu)$. First note that $S(\mu)$ is closed and $V=R^2$ so if $S(\mu)$ were not discrete, $S(\mu)$ would be conjugate to the orthogonal group. Then by Theorem 4, B would have conjugate complex eigenvalues. Hence $S(\mu)$ is discrete, and $B \in E_c(\mu)$ by Proposition 1. Now suppose $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S(\mu)$. Then since $B \in E_c(\mu)$ $BD = DB$ and so $c = b = 0$. Since $S(\mu)$ is a compact group, the fact that $D^n \in S(\mu)$ for all n shows $|a| = |d| = 1$. A direct computation now shows that $S(\mu) = S(M) = \{\pm I, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\}$.

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